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Note

Requiring chords in cycles

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Abstract

Jamison proved that every cycle of length greater than three in a graph has a chord—in other words, the graph is chordal—if and only if every k-cycle is the sum of $k-2$ triangles. This result generalizes to having or not having crossing chords and to having strong chords, with similar characterizations of a variety of graph classes that includes chordal bipartite, distance-hereditary, and strongly chordal graphs.

Keywords: Chord; Chordal graph; Chordal bipartite graph; Distance-hereditary graph; Strongly chordal graph

1. Generalizing a theorem of Jamison

A *chord* is an edge joining two nonconsecutive vertices along a cycle of length of at least 4. The *sum of cycles* means the symmetric difference (or 'ring-sum,' denoted by $\oplus$) of their edge sets. For instance, in the graph in Fig. 1 the 4-cycle with edge set $\{a,b,c,d\}$ is the sum $\{a,e,f\} \oplus \{b,f,g\} \oplus \{c,g,h\} \oplus \{d,e,h\}$ of 3-cycles, while the 4-cycle $\{a,b,g,e\}$ is the sum $\{a,e,f\} \oplus \{b,g,f\}$.

Theorem 1 will show, in general, how the existence of chords in larger cycles is related to the ability to write cycles as sums of specific sizes of smaller cycles. The theorem's corollaries will then give specific instances of interest.

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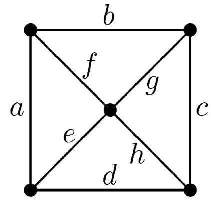

Fig. 1. An example illustrating sums of cycles.

Theorem 1. *For every graph G and $n \geqslant 3$, the following are equivalent*:

(1a) *Every cycle in G with length greater than n has a chord.*
(1b) *Every k-cycle in G with $k \geqslant 3$ is the sum of a set of cycles that consists of c_i distinct i-cycles for each $i \leqslant n$ such that*

$$\sum_{3 \leqslant i \leqslant n} (i-2)c_i = k-2. \tag{1}$$

Proof. First suppose $n \geqslant 3$, condition (1a) holds, and C is a k-cycle with $k \geqslant 3$. If $k \leqslant n$, then C is trivially the sum of itself alone and equality (1) holds with $c_k = 1$ and $c_i = 0$ whenever $i \neq k$. So suppose $k > n$ and, by (1a), C has chord e; say $C = C_a \oplus C_b$, where $C_a \cap C_b = \{e\}$, $|C_a| = a$, $|C_b| = b$, and $a + b = k + 2$. Arguing inductively on k, using $3 \leqslant k \leqslant n$ as basis, assume that C_a is the sum of a_i distinct i-cycles for each $i \leqslant n$ such that

$$\sum_{3 \leqslant i \leqslant n} (i-2)a_i = a-2,$$

and C_b is the sum of b_i distinct i-cycles for each $i \leqslant n$ such that

$$\sum_{3 \leqslant i \leqslant n} (i-2)b_i = b-2.$$

Then C is the sum of $c_i = a_i + b_i$ distinct i-cycles for each $i \leqslant n$ such that

$$\sum_{3 \leqslant i \leqslant n} (i-2)c_i = a+b-4 = (k+2)-4 = k-2.$$

Conversely, suppose $k > n \geqslant 3$ and C is a k-cycle that is the sum of a set $\mathscr{S}$ of cycles that consists of c_i distinct i-cycles for each $i \leqslant n$ such that equality (1) holds. If, for each $i \leqslant n$, each of those c_i distinct i-cycles were to have at most $i-2$ edges in common with C, then

$$|E(C)| = k \leqslant \sum_{3 \leqslant i \leqslant n} (i-2)c_i = k-2.$$

So at least one i-cycle in $\mathscr{S}$ must have at least $i-1$ edges in common with C—in fact, exactly $i-1$, since $k > n$ implies that such a cycle cannot have all i edges in C. (Note that there must actually be at least two cycles in $\mathscr{S}$, each of which has all but one of its edges in C in order to avoid the $k \leqslant k-2$ contradiction above—this will be used in the proof of Lemma 1 in Section 2.) The one remaining edge in such a cycle will be a chord of C. $\square$

A graph is *chordal* [10] if every k-cycle, $k \geqslant 4$, has a chord—in other words, every cycle large enough to have a chord does have a chord.

Corollary 1 (*Jamison [7]*). *A graph G is chordal if and only if every k-cycle, $k \geqslant 3$, is the sum of $k-2$ distinct 3-cycles.*

Proof. Take $n = 3$ in Theorem 1. Equality (1) becomes $c_3 = k - 2$. $\square$

A *hole* is an induced cycle of length at least 4, and a *long hole* [2,3,5] is a hole of length at least 5. (A graph is chordal if and only if it has no holes.)

Corollary 2. *A graph has no long holes if and only if every k-cycle, $k \geqslant 3$, is the sum of c_3 distinct 3-cycles and c_4 distinct 4-cycles such that $c_3 + 2c_4 = k - 2$.*

Proof. Take $n = 4$ in Theorem 1. $\square$

A graph is *chordal bipartite* [10] if it is bipartite and every cycle of length at least 6 has a chord—in other words, every cycle in the bipartite graph large enough to have a chord does have a chord. (A graph is chordal bipartite if and only if it is bipartite and has no long holes.)

Corollary 3 (*McKee [8]*). *A bipartite graph is chordal bipartite if and only if every k-cycle, $k \geqslant 3$, is the sum of $(k/2) - 1$ distinct 4-cycles.*

Proof. This is a special case of Corollary 2, since G being bipartite implies that $c_3 + 2c_4 = 0 + 2c_4 = k - 2$. $\square$

2. Crossing chords and uniqueness

Two chords ab and $a'b'$ of a cycle C are *crossing chords* if their endpoints come in the order a, a', b, b' around C. Theorems 2 and 3 will relate the existence of crossing chords in larger cycles to the ability to write cycles as the sum of smaller cycles. Again, corollaries will give specific instances of interest.

Lemma 1. *Suppose C is a k-cycle in graph G with $k > n \geqslant 3$, no i-cycle in G with $3 \leqslant i \leqslant n$ has a chord, and C is the sum of a set $\mathscr{S}$ of cycles that consists of c_i distinct i-cycles for each $i \leqslant n$ such that equality (1) holds. Then there will be at least two cycles in $\mathscr{S}$, each of which has all but one of its edges in C, and every edge of each cycle in $\mathscr{S}$ is either an edge of C or a chord of C.*

Proof. Suppose C and $\mathscr{S}$ are as in the statement of the lemma and argue by induction on $k \geqslant 4$. If $k = 4$, then $n = 3$ (since $k > n \geqslant 3$), equality (1) reduces to $c_3 = 2$, and $\mathscr{S}$ consists of two 3-cycles, each consisting of two edges of C and one chord of C. So suppose $k > 4$. As in the proof of the (1b) $\Rightarrow$ (1a) direction of Theorem 1, there will be at least two cycles in $\mathscr{S}$, each of which has all but one of its edges in C with the remaining edge a chord of C. Let C' be any one of those cycles, say with $j-1$ edges in C and the one remaining edge e' a chord of C. Consider the cycle $C^* = C \oplus C'$ of length $k^* = k - j + 2 < k$. If $k^* \leqslant n$, then the assumption that no i-cycle in G with $3 \leqslant i \leqslant n$ has a chord shows that C' and C^* must be the two cycles in $\mathscr{S}$, each consisting of edges of C together with the one chord e' of C. Otherwise, $k^* > n$ and so, by the inductive hypothesis, two of the cycles in $\mathscr{S} - \{C'\}$ will have all but one edge in C^*. Thus, one of those two will avoid containing e' and will, like C', have all but one of its edges in C; all non-C edges of all the cycles in $\mathscr{S}$ will be chords of C. $\square$

Theorem 2. *For every graph G and $n \geqslant 3$ such that no i-cycle in G with $3 \leqslant i \leqslant n$ has a chord, the following are equivalent*:

(2a) *Every cycle in G with length greater than n has a chord but no crossing chords*.
(2b) *Every k-cycle in G with $k \geqslant 3$ is the sum of a unique set of cycles that consists of c_i distinct i-cycles for each $i \leqslant n$ such that*

$$\sum_{3 \leqslant i \leqslant n} (i-2)c_i = k - 2. \tag{2}$$

Proof. Suppose $n \geqslant 3$ and no i-cycle in G with $3 \leqslant i \leqslant n$ has a chord.

First, suppose condition (2a) holds. Suppose $k \geqslant 3$ and C is a k-cycle of G. If $k \leqslant n$, then C is trivially the sum of itself alone and equality (2) holds with $c_k = 1$ and $c_i = 0$ whenever $i \neq k$. For the uniqueness of this singleton set $\{C\}$, suppose C is the sum of c_i distinct i-cycles for each $i \leqslant n$ such that equality (2) holds. If, for each $i \leqslant n$, each of those c_i distinct i-cycles were to have at most $i-2$ edges in common with C, then the $k \leqslant k-2$ contradiction in the proof of Theorem 1 would occur (by the same argument as there). So at least one of those i-cycles must have at least $i-1$ edges in common with C—but not exactly $i-1$, since C has no chord. Thus that i-cycle must have all i edges in common with C, making $i = k$ and $c_i = c_k = 1$.

So suppose $k > n$ and, by (2a), C has a chord e with no crossing chords; say $C = C_a \oplus C_b$, where $C_a \cap C_b = \{e\}$, $|C_a| = a$, $|C_b| = b$, and $a + b = k + 2$. Arguing inductively on k, using $3 \leqslant k \leqslant n$ as the basis, assume that C_a is uniquely the sum of a_i distinct i-cycles for each $i \leqslant n$ such that

$$\sum_{3 \leqslant i \leqslant n} (i-2)a_i = a - 2,$$

and C_b is uniquely the sum of b_i distinct i-cycles for each $3 \leqslant i \leqslant n$ such that

$$\sum_{3 \leqslant i \leqslant n} (i-2)b_i = b - 2.$$

Then C is the sum of these $c_i = a_i + b_i$ distinct i-cycles for each $i \leqslant n$ and

$$\sum_{3 \leqslant i \leqslant n} (i-2)c_i = a + b - 4 = (k+2) - 4 = k - 2.$$

For the uniqueness of this set of cycles, note that each edge of each of the cycles in any such set $\mathscr{S}$ will be an edge of C or a chord of C by Lemma 1. Since C has no chords that cross e, the uniqueness of $\mathscr{S}$ follows from applying the inductive hypothesis to C_a and C_b.

Conversely, suppose condition (2b) holds and C is a k cycle with $k > n$. By Theorem 1, C has a chord e. Suppose e were to have a crossing chord f of C, arguing toward a contradiction. Say $C = C_a \oplus C_b$, where $C_a \cap C_b = \{e\}$, and also $C = C_{a'} \oplus C_{b'}$, where $C_{a'} \cap C_{b'} = \{f\}$. Applying (2b) to C_a and C_b would produce a set cycles summing to C that consisted of c_i distinct i-cycles for each $i \leqslant n$ such that—as in the (1a) $\Rightarrow$ (1b) direction of the proof of Theorem 1—equality (2) holds; note that two of these i-cycles would contain e and none would contain the crossing chord f. In the same way, there would exist a similar set of cycles, two containing f and none containing e. But these two sets would contradict the uniqueness of the set of i-cycles for C in (2b). $\square$

To show why the 'no i-cycle in G with $3 \leqslant i \leqslant n$ has a chord' assumption is needed in Theorem 2, let G consist of a 4-cycle C together with one diagonal. This G would vacuously satisfy condition (2a) with $n = 4$, but not (2b)—C would be both the sum of two 3-cycles and of one 4-cycle, both sets satisfying equality (2) (since $2(3-2) = 4-2$ and $1(4-2) = 4-2$, respectively).

The widely studied class of 2-*trees* is defined inductively, starting from K_2, as follows: if G is a 2-tree with $e \in E(G)$ and if $H \cong K_3$ is vertex disjoint from G with $e' \in E(H)$, then the graph formed from G and H by identifying edges e and e' (along with their endpoints) is also a 2-tree.

Corollary 4 (*McKee [9]*). *A* 2*-connected graph is a* 2*-tree if and only if every k-cycle, $k \geqslant 3$, is the sum of a unique set of $k-2$ distinct* 3*-cycles*.

Proof. Take $n = 3$ in Theorem 2 and use Corollary 1 and that a 2-connected graph is a 2-tree if and only if it is chordal and no cycle has crossing chords. (The latter follows by, for instance, Theorem 1.1 of [11].) $\square$

Corollary 3 could be used similarly to characterize the 2*-*trees* introduced and motivated in [9]—these chordal bipartite analogs of 2-trees are defined just as 2-trees were above, except with '$H \cong K_3$' replaced by '$H \cong K_{2,h}(h > 2)$' (noting that identifying e and e' can result in two nonisomorphic graphs in this case). A 2-connected bipartite graph is a $2*$-tree if and only if every k-cycle, $k \geqslant 3$, is the sum of a unique set of $(k/2) - 1$ distinct 4-cycles.

Theorem 3 shows an alternative to Theorem 2, replacing there being a *unique* set of cycles satisfying equality (2) with there being *several* (meaning at least two) such sets.

Theorem 3. *For every graph G and $n \geqslant 3$ such that no i-cycle in G with $3 \leqslant i \leqslant n$ has a chord, the following are equivalent*:

(3a) *Every cycle in G with length greater than n has crossing chords.*
(3b) *Every k-cycle in G with $k > n$ is the sum of several sets of cycles that consists of c_i distinct i-cycles for each $i \leqslant n$ such that*

$$\sum_{3 \leqslant i \leqslant n} (i-2)c_i = k-2. \tag{3}$$

Proof. First suppose $k > n \geqslant 3$ and condition (3a) holds; say e and f are crossing chords in the k-cycle C. By Theorem 1, at least one set of cycles as in (3b) exists. Indeed, by the proof of Theorem 1, there is such a set of cycles with two of them containing e and none of them containing the crossing chord f, and there is such a set of cycles with two of them containing f and none of them containing the crossing chord e. Hence, there are at least two such sets of cycles.

Conversely, suppose $k > n \geqslant 3$ and that condition (3b) holds; say C is a k-cycle with $k \geqslant 4$, and $\mathscr{S}_1$ and $\mathscr{S}_2$ are two different sets of i-cycles as in (3b). Condition (3a) follows from the assumption that no i-cycle in G with $3 \leqslant i \leqslant n$ has a chord and Lemma 1. □

To show why the 'no i-cycle in G with $3 \leqslant i \leqslant n$ has a chord' assumption is needed in Theorem 3, let G consist of a 5-cycle C together with two diagonals, both incident with a common vertex of C. This G would satisfy condition (3b) with $n = 5$—C would be the sum of one 3-cycle and one 4-cycle and also of three 3-cycles, both satisfying equality (3) (since $1(3-2) + 1(4-2) = 5-2$ and $3(3-2) = 5-2$, respectively)—but not (3a).

A graph is a *block graph*, sometimes called a *Husimi tree*, if it is connected and every *block* (maximal 2-connected induced subgraph) is complete.

Corollary 5. *A connected graph is a block graph if and only if every k-cycle, $k \geqslant 3$, is the sum of several sets of $k-2$ distinct 3-cycles.*

Proof. Take $n = 3$ in Theorem 3 and use that a graph is a block graph if and only if every cycle of length at least 4 has crossing chords. □

A graph G is a *distance-hereditary graph* [1,6] whenever the distance between two vertices in a connected induced subgraph always equals their distance in the full graph. Bipartite distance-heredity graphs are also studied in [1].

Corollary 6. *A bipartite graph is distance-hereditary if and only if every k-cycle, $k \geqslant 4$, is the sum of several sets of $(k/2) - 1$ distinct 4-cycles.*

Proof. Take $n=4$ in Theorem 3 and use that, from [6], a bipartite graph is distance-hereditary if and only if every cycle of length at least 6 has crossing chords. □

3. Strong chords and even cycles

An *even cycle* is a cycle of even length. A *strong chord* [4] of a cycle C is a chord e such that at least one of the two paths within C that form cycles with e forms an even cyclewith e.

Note that, in an odd cycle, every chord is trivially a strong chord; in an even cycle C, e is a strong chord if and only if $C = C_a \oplus C_b$, where $C_a \cap C_b = \{e\}$, and C_a and C_b are both even cycles.

Theorem 4. *For every graph G and $n \geqslant 4$, the following are equivalent*:

(4a) *Every cycle in G with length greater than n has a strong chord.*
(4b) *Every k-cycle in G with $k \geqslant 4$ is the sum of a set of even cycles that consists of c_i distinct i-cycles for each $i \leqslant n$ such that*

$$\sum_{4 \leqslant i \leqslant n} (i-2)c_i = k-2. \tag{4}$$

Proof. The proof is almost identical to the proof of Theorem 1, except restricting k and i to be even, and replacing 'cycles' with 'even cycles' and 'chord' with 'strong chord' throughout (noting that a and b will also be even in the proof). $\square$

A graph is *strongly chordal* if it is chordal and every cycle of length at least 5—or, equivalently, every even cycle of length at least 6—has a strong chord; see [4,10].

Corollary 7. *A chordal graph G is strongly chordal if and only if, for every even $k \geqslant 4$, every k-cycle in G is the sum of* $(k/2) - 1$ *distinct* 4-*cycles*.

Proof. This is a special case of taking $n = 4$ in Theorem 4. $\square$

Corollary 8. *A graph G is strongly chordal if and only if every k-cycle, $k \geqslant 3$, is the sum of* $k-2$ *distinct* 3-*cycles and, when k is even, is also the sum of* $(k/2)-1$ *distinct* 4-*cycles*.

Proof. This is a combined version of Corollaries 1 and 7. $\square$

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